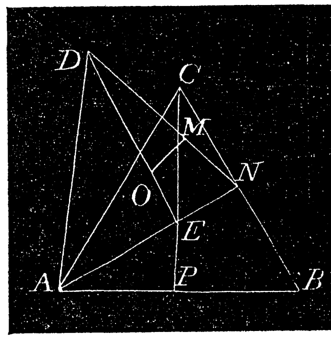
SOLUTION. Put $2a =$ the linear edge, $b =$ the whole superficies and c $=$ the solid contents of the tetrahedron formed by the four boards; and let r and R represent the radii of its inscribed and circumscribed spheres:

Then is $2a = 2r\sqrt{6}$; $b = 24r^2\sqrt{3}$; $c = 8r^3\sqrt{3}$, and $3r = R$. For. let ABC represent the base of the tetrahedron and draw the perpendiculars AN and CP bisecting AB and BC in P and N. At the intersection of AN and CP make ED perpendicular to AN, and suppose it also perpendicular to the plane of the paper. Make $AD = AC$; then ED will be the altitude of the tetrahedron, and ADN, a vertical section through one of its edges and its axis. Make $NM = NE$, and draw MO perpendicular to ND; then will O be the common center of r and R. Because $AC = 2a$, $\therefore$ $PC = AN = DN = a\sqrt{3}$. $PC : PB :: NC : NE = \frac{1}{3}a\sqrt{3} = EP$. $\therefore$ $AE = \frac{2}{3}a\sqrt{3}$, and $DE = \frac{2}{3}a\sqrt{6}$. Also, by similar triangles, $DN : NE :: DO\ (= ED - r) : OM = OE = r = \frac{1}{6}a\sqrt{6}$. $\therefore$ $a = r\sqrt{6}$. . . . (A)



This shows that $OE = \frac{1}{4}ED$ and that $3r = R$.

Again, $4(AP \times CP) = b = 4a^2\sqrt{3}$; but $a = r\sqrt{6}$, $\therefore$ $b = 24r^2\sqrt{3}$. . . . (B)

Lastly, $\frac{1}{3}(DE \times AP \times CP) = c = \frac{1}{3}(2a^3\sqrt{3}) = 8r^3\sqrt{3}$. . . . . (C)

If $2a' =$ the linear edge of a tetrahedron whose angles are at the centers of the four balls when in position touching each other; then is $a' = 3$ inches. Hence, as the boards are 1 inch thick, $OE = \frac{1}{2}\sqrt{6} + 3 + 1 = r$, and, from (A), we have $a = 3 + 4\sqrt{6}$. And $2a_1 =$ the linear edge of the inside of the box, $a_1 = 3 + 3\sqrt{6}$.

From (A) we have $r = \frac{1}{6}a\sqrt{6} = \frac{1}{2}\sqrt{6} + 4$. Substituting this value of r in (B) we get $b = 7.880242$ square feet. Substituting the two values of r, corresponding to a and a_1, in (C), we find for the difference of the two tetrahedrons, or the solidity of the box, 931.4226928 cubic inches.

Because $2a = 25.5959176$ inches, $= e$; $\therefore$ $r = \frac{1}{12}e\sqrt{6}$; $R = \frac{1}{4}e\sqrt{6}$; $b = e^2\sqrt{3}$, and $c = \frac{1}{12}e^2\sqrt{2}$.

SOLUTION OF PROBLEMS IN NO. FIVE, VOL. III.

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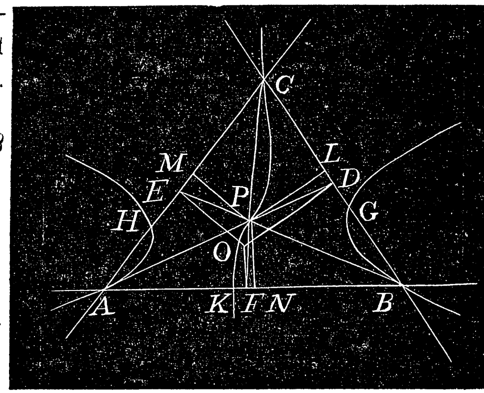
129. "Three lines through the vertices of a triangle meet in a point, P. Through the intersection of each with the opposite side a perpendicular to that side is drawn and these three perpendiculars meet in a point. Find the locus of P."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

LET D, E, F be the points of intersection of the three lines with the sides of the triangle, O the common point of the perpendiculars at D, E, F. Draw PL, PM, PN perpendicular to BC, CA, AB. Put $PL = \alpha$, $PM = \beta$, $PN = \gamma$. Then we easily find $BD = \dfrac{ac\gamma}{a\alpha + b\beta}$



$DC = \dfrac{ab\beta}{b\beta + c\gamma}$, $CE = \dfrac{ba\alpha}{c\gamma + a\alpha}$, $EA = \dfrac{bc\gamma}{c\gamma + a\alpha}$, $AF = \dfrac{cb\beta}{a\alpha + b\beta}$, $FB = \dfrac{ca\alpha}{a\alpha + b\beta}$; and by trigonometry we have

$$OF = AE \operatorname{cosec} A - AF \cot A = BD \operatorname{cosec} B - BF \cot B,$$

whence $\quad BC(DC - BD) + CA(EA - CE) + AB(FB - AF) = 0.$

By substitution we have

$$a^2\left(\frac{b\beta - c\gamma}{b\beta + c\gamma}\right) + b^2\left(\frac{c\gamma - a\alpha}{c\gamma + a\gamma}\right) + c^2\left(\frac{a\alpha - b\gamma}{a\alpha + b\beta}\right) = 0,$$

whence by reduction we find

$a^2\alpha^2(\beta \cos B - \gamma \cos C) + b^2\beta^2(\gamma \cos C - \alpha \cos A) + c^2\gamma^2(\alpha \cos A - \beta \cos B) = 0 \,..(1)$ the trilinear equation to the locus of P, which is, therefore, a line of the third order.

By tracing the curve we find that it consists of three disconnected branches. From (1) and $a\alpha + b\beta + c\gamma = 2\triangle$ . . . (2), by making $\gamma = 0$, we find $\alpha = 0$, $c \sin B$, or $b \cos A \sin B$, and $\beta = c \sin A$, 0, or $a \sin A \cos B$; hence the three branches pass respectively through B, A and K, and $AK = a \cos B$.

In a similar manner we find that the branch through K passes through C, that the branches through A and B intersect AC and BC in H and G, and that $AH = a \cos C$, and $BG = b \cos C$. The branch that passes through the vertex opposite the longest side intersects that side, but the other branches do not intersect the sides opposite the vertices through which they pass.

The values $a\alpha = \frac{2}{3}\triangle$, $b\beta = \frac{2}{3}\triangle$, $c\gamma = \frac{2}{3}\triangle$ satisfy (1); hence the curve passes througnh the center of gravity of the triangle. The values $\alpha = 2R \times \cos B \cos C$, $\beta = 2R \cos A \cos C$, $\gamma = 2R \cos A \cos B$ satisfy (1); hence the curve passes through the intersection of the three perpendiculars.

If $a = b$, (1) becomes $(\alpha - \beta)[a^2c\alpha\beta - a(2a^2 - c^2)(\alpha\gamma + \beta\gamma) + c^3\gamma^2] = 0$, or $\alpha - \beta = 0$ . . . (3), and $a^2c\alpha\beta - a(2a^2 - c^2)(\alpha\gamma + \beta\gamma) + c^3\gamma^2 = 0$. . . .(4)

Equation (3) represents the line which bisects the angle ACB, and eq. (4) represents two opposite hyperbolas, passing through A and B, and having the transverse axis parallel to AB. If $a = b = c$, (1) becomes $(\alpha - \beta) \times (\beta - \gamma)(\gamma - \alpha) = 0$, which is the equation to the bisectors of the angles of the triangle.

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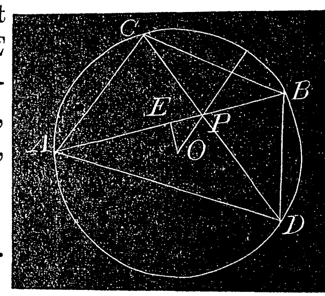
134 "Through a point taken at random in the surface of a circle, two chords are drawn, one at random and the other at right angles to the radius through that point; find the average area of the quadrilateral formed by joining the extremities of the chords."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

LET O be the center of the circle, P the random point in its surface, AB the random chord and CD the chord at right angles to the radius through P. Draw OE perpendicular to AB. Let $OP = x$, and angle $APO = \varphi$; then $\angle\ APC = \frac{1}{2}\pi - \varphi$, $CD = 2\sqrt{(r^2 - x^2)}$, $AB = 2\sqrt{(r^2 - x^2\sin^2\varphi)}$, and the area of the quadrilateral $ABCD = AB \times CD \times \frac{1}{2} \sin APC = 2\sqrt{(r^2 - x^2)} \times \sqrt{(r^2 - x^2 \sin^2 \varphi)} \cos \varphi$.



If $\varDelta$ be the average area required, we have

$$\varDelta = \frac{2\int_0^r \int_0^{\frac{1}{2}\pi} (r^2 - x^2)^{\frac{1}{2}} (r^2 - x^2\sin^2\varphi)^{\frac{1}{2}} . 2\pi x dx d\varphi}{\int_0^r \int_0^{\frac{1}{2}\pi} 2\pi x dx d\varphi}$$

$$= \frac{8}{\pi r^2} \int_0^r \int_0^{\frac{1}{2}\pi} (r^2 - x^2)^{\frac{1}{2}} (r^2 - x^2\sin^2\varphi)^{\frac{1}{2}} \cos\varphi\ x dx\ d\varphi,$$

$$= \frac{4}{\pi r^2} \int_0^r (r^2 - x^2) x dx + \frac{4}{\pi} \int_0^r (r^2 - x^2)^{\frac{1}{2}} \sin^{-1}\left(\frac{x}{r}\right) dx,$$

$$= \frac{r^2}{\pi} + \frac{1}{\pi} \left[2x(r^2 - x^2)^{\frac{1}{2}} \sin^{-1}\frac{x}{r} + r^2\left(\sin^{-1}\frac{x}{r}\right)^2 - x^2 \right]_0^r$$

$$= \tfrac{1}{4}\pi r^2.$$

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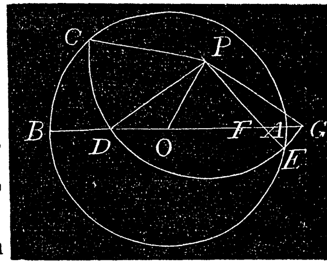
135. "A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle. Find the chance that the line intersects a given diameter."

SOLUTION BY HENRY HEATON B. S., DES MOINES, IOWA.



LET ABC be the given circle, O its center, AB the given diameter, and P the point taken at random.

Put $OP = x$ and angle $AOP = \theta$. With P as center and radius $= OA = 1$, describe the arc $CDEG$ cutting BA, or BA produced, in D and G.

If we suppose P fixed, and θ less than $\cos^{-1}\frac{1}{2}x$, G is outside the circle, and the required probability is $DE \div CDE = [\theta - \sin^{-1}(x\sin\theta) + \cos^{-1}\frac{1}{2}x] \div 2\cos^{-1}\frac{1}{2}x$. But if θ is greater than $\cos^{-1}\frac{1}{2}x$ and less than $\frac{1}{2}\pi$, G is within the circle and the required probability is $DG \div CDGE = \cos^{-1}(x\sin\theta) \div \cos^{-1}\frac{1}{2}x$.

The probability that x and θ will have any particular value less than 1 and $\frac{1}{2}\pi$, respectively, is $xdx\,d\theta \div \frac{1}{4}\pi$. Hence the required probability is

$$\frac{2}{\pi}\int_0^1\int_0^u \frac{[\theta - \sin^{-1}(x\sin\theta) + \cos^{-1}\frac{1}{2}x]xdxd\theta}{\cos^{-1}\frac{1}{2}x} + \frac{4}{\pi}\int_0^1\int_u^{\frac{1}{2}\pi}\frac{\cos^{-1}(x\sin\theta)xdxd\theta}{\cos^{-1}\frac{1}{2}x}$$

$$= \frac{1}{2} - \frac{3\sqrt{3}}{4\pi} + \frac{2}{\pi}\int_0^1\int_0^{\frac{\pi}{3}}\left(\frac{\cos^{-1}(x\sin\theta)}{\cos^{-1}\frac{1}{2}x}\right)xdxd\theta + \frac{2}{\pi}\int_0^1\int_u^{\frac{\pi}{2}}\left(\frac{\cos^{-1}(x\sin\theta)}{\cos^{-1}\frac{1}{2}x}\right)xdxd\theta$$

in which $u = \cos^{-1}\frac{1}{2}x$. This integral, I think, can only be found by approximate methods.

[The foregoing solution is published instead of the solution by Mr. Seitz, which we promised, in No. 6, Vol. III, to publish in this No. We have made the substitution because the method pursued by Mr. Seitz in his solution was subsequently found to be defective.

It will be seen that, in the above solution, the required probability is obtained by taking the *sum* of the probabilities for the different points.

In the solution obtained by Mr. Seitz, and in which a finite integration is effected, the sum of the intersecting lines from all the points is taken for the numerator, and the sum of all the lines from all the points, for the denominator of the fraction representing the required probability. If the whole number of lines that can be drawn from each point were equal, this method would be correct, but as that is not the case, the probability of drawing the different lines is not the same, and hence the separate probabilities for the different points must be summed for the required probability.

This peculiarity of the question was not observed in our first examina-ination of the solutions, and was, subseqaently, pointed out by Mr. Heaton, to whom the manuscript solutions were submitted for examination and comparison.—Ed.]

136. "Suppose a planked floor with thin visible seams between the planks. Let there be a thin straight rod not so long as the breadth of the planks. This rod, being tossed up at hazard, will either fall quite clear of the seams or will lie across one seam. Prove that in the long run the fraction of the whole number of trials in which a seam is intersected, will be the fraction which twice the length of the rod is to the circumference of the circle having the breadth of the plank for its diameter."

SOLUTION BY WILLIAM HOOVER, BELLEFONTAINE, OHIO.

Let $2a$ = width of planks, $2b$ = length of rod, φ = the angle the rod makes with a perpendicular to the seams, and let x = the distance of the center of the rod from the nearest seam. The rod will cross a seam for all values of x from 0 to $b\cos\varphi$; hence the chance of crossing the seam is $4\varphi \div 2\pi$, and as the chance that the center of the rod will take the particular position at the distance x from the seam is $dx \div a$, the required probability will evidently be

$$\frac{2}{\pi a}\int \varphi dx = \frac{2b}{\pi a}\int_0^{\frac{1}{2}\pi} \varphi \sin\varphi \, d\varphi = \frac{2b}{\pi a}.$$

NOTE, BY CHRISTINE LADD, UNION SPRINGS, N. Y.—THE relation between the sides and diagonals of the contra parallelogram given in problem 118 is a particular case of the relation between the two values of the side of a triangle when the remaining sides and the angle opposite one of them are given. If c be the side required and B the angle given, we have

$$c = a \cos B = b \cos a,$$
$$cc' = a^2 \cos^2 B - b^2 \cos^2 a = a^2 - b^2.$$

SOLUTION OF PROBLEMS IN NO. SIX, VOL. III.

SOLUTIONS of problems in No. 6, Vol. III, have been received as follows: From Marcus Baker, 137; Henry Gunder, 137 and 139; H. Heaton, 139, 140 and answer to query; Artemas Martin, 139 and 140; W. L. Marcy, 137 and 140; Prof. J. Scheffer, 138; E. B. Seitz, 137 and 140; Prof. D. Trowbridge, 140; R. J. Adcock, answer to query.

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137. "A point D, is given in position between two lines which make a given angle at A. Find the position of a given line, BC, drawn through D, and intersecting the two lines in the points B and C."